# Bessel Eqn

- Has the formula $x^2y'' + xy' + (x^2 - v^2)y = 0$ where $v$ is a constant called the order of the Bessel eqn or index.

- Bessel eqn of order zero:

This occurs when $v=0$.
The eqn is now $x^2y'' + xy' + x^2y = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $x^2y'' + xy' + x^2y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} +$$

$$\sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

We want all summations to have $x^{n+r}$.

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} +$$

$$\sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

Take $n=0$:

$a_0(r)(r-1)+a_0(r)=0$

$a_0(r^2-r+r)=0$

$r^2=0$

$r_1=r_2=0$

For $r=0$, take $n=1$:

$a_1(1)(1-1)+a_1(1)=0$

$a_1=0$

For $r=0$, take $n\geq 2$:

$a_n(n+r)(n+r-1)+a_n(n+r)+a_{n-2}=0$

$a_n((n+r)(n+r-1)+(n+r))=-a_{n-2}$

Let $a=n+r$

$a(a-1)+a$

$a^2-a+a$

$a^2$

$(n+r)^2$

$$a_n=\frac{-a_{n-2}}{(n+r)^2}$$

$$=\frac{-a_{n-2}}{n^2} \quad \text{Since } r=0$$

Note: Since $a_1=0$, $a_3, a_5, \ldots, a_{2k+1}=0$.

| $n=2$: | $n=4$: | $n=6$: |
|---|---|---|
| $a_2=\frac{-a_0}{2^2}=\frac{-1}{4}$ | $a_4=\frac{-a_2}{4^2}=\frac{1}{2^2\cdot 4^2}$ | $a_6=\frac{-a_4}{6^2}=\frac{-1}{2^2\cdot 4^2\cdot 6^2}$ |

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \dots \quad r=0$$
$$= 1 - \frac{x^2}{4} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

$y_1$ is called the Bessel function of the first kind of order zero and the standard notation is $J_0(x)$.

To find $y_2$, we need to use the Frobenius Method. We know that $r_1 = r_2$.

$$y_2 = \log(x)\, y_1 + \sum_{n=0}^{\infty} a_n'(r)\, x^{n+r} \Big|_{r=r_1}$$

$$a_2' = \partial_r(a_2)$$
$$= \partial_r\left(\frac{-a_0}{(2+r)^2}\right)$$
$$= \partial_r\left(\frac{-1}{(r+2)^2}\right)$$
$$= \frac{2(r+2)}{(r+2)^4}$$
$$= \frac{2}{(r+2)^3}$$

$$a_2' x^{2+r} \Big|_{r=0} = \frac{x^2}{4}$$

$$a_4' = \partial_r(a_4)$$
$$= \partial_r\left(\frac{-a_2}{(4+r)^2}\right)$$
$$= \left(\frac{1}{(2+r)^2(4+r)^2}\right)'$$
$$= \left(\frac{1}{(r^2+6r+8)^2}\right)'$$
$$= \frac{-2(r^2+6r+8)(2r+6)}{(r^2+6r+8)^4}$$
$$= \frac{-2(2r+6)}{(r^2+6r+8)^3}$$

$$a_4' x^{4+r} \Big|_{r=0} = \frac{-12x^4}{512}$$

$$y_2 = \log(x)\, y_1 + \frac{x^2}{4} - \frac{12}{512}x^4 + \dots$$

$y_2$ is called the Bessel function of the second kind of order zero and the standard notation is $Y_0(x)$.

The general soln is $y = C_1 J_0(x) + C_2 Y_0(x)$.

– Bessel eqn of order One-Half:

This occurs when $\nu = \frac{1}{2}$.

The eqn is now $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$.

We can rewrite $x^2y'' + xy' + x^2y - \frac{1}{4}y = 0$ as

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} +$$

$$\sum_{n=2}^{\infty} a_{n-2}x^{n+r} - \sum_{n=0}^{\infty} \frac{1}{4}a_n x^{n+r} = 0$$

Take $n=0$:

$a_0(r)(r-1) + a_0(r) - \frac{1}{4}a_0 = 0$

$a_0(r^2 - r + r - \frac{1}{4}) = 0$

$r^2 = \frac{1}{4}$

$r_1 = \frac{1}{2}, \quad r_2 = -\frac{1}{2}$

For $r = \frac{1}{2}$, take $n=1$:

$a_1(\frac{3}{2})(\frac{1}{2}) + a_1(\frac{3}{2}) - \frac{1}{4}a_1 = 0$

$\frac{3a_1}{4} + \frac{3a_1}{2} - \frac{a_1}{4} = 0$

$2a_1 = 0$

$a_1 = 0$

For $r = \frac{1}{2}$, take $n \geq 2$:

$a_n (n+r)(n+r-1) + a_n (n+r) + a_{n-2} - \frac{1}{4} a_n = 0$

$a_n ((n+r)(n+r-1) + (n+r) - \frac{1}{4}) = -a_{n-2}$

Let $a = n+r$

$a(a-1) + a - \frac{1}{4}$

$a^2 - a + a - \frac{1}{4}$

$a^2 - \frac{1}{4}$

$(a - \frac{1}{2})(a + \frac{1}{2})$

$(n+r - \frac{1}{2})(n+r + \frac{1}{2})$

$$a_n = \frac{-a_{n-2}}{(n+r-\frac{1}{2})(n+r+\frac{1}{2})}$$

$$= \frac{-a_{n-2}}{(n)(n+1)}$$

Note: Since $a_1 = 0$, $a_3, a_5, \ldots, a_{2k+1} = 0$

| | | |
|---|---|---|
| $a_2 = \frac{-a_0}{(2)(3)}$ | $a_4 = \frac{-a_2}{(4)(5)}$ | $a_6 = \frac{-a_4}{(6)(7)}$ |
| $= \frac{-1}{6}$ | $= \frac{1}{120}$ | $= \frac{-1}{5040}$ |

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^{1/2} + a_2 x^{5/2} + a_4 x^{9/2} + a_6 x^{13/2} + \ldots$$

$$= x^{1/2} - \frac{x^{5/2}}{6} + \frac{x^{9/2}}{120} - \frac{x^{13/2}}{5040} + \ldots$$

$y_1$ is called the Bessel function of the first kind of order one-half. The standard notation is $J_{\frac{1}{2}}(x)$.

To find $y_2$, we need to use the Frobenius Method.

$$r_1 - r_2 = \tfrac{1}{2} - (-\tfrac{1}{2})$$
$$= 1$$

$N = 1$

$$y_2 = a\log(x)\, y_1 + x^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r)\, x^n\right)$$

$$a = \lim_{r \to r_2} (r - r_2)\, a_N(r)$$
$$= \lim_{r \to -\frac{1}{2}} (r + \tfrac{1}{2})\, a_1(r)$$

$$a_1(1+r)(r) + a_1(1+r) - \tfrac{1}{4} a_1 = 0$$
$$a_1\left((1+r)r + 1 + r - \tfrac{1}{4}\right) = 0$$
$$a_1\left(r^2 + 2r + 3/4\right) = 0$$

Since we have $r$ approaching $-\frac{1}{2}$, $r^2 + 2r + 3/4 \neq 0$.
Hence, this means that $a_1 = 0$.
Hence, $a = 0$.

$$c_n = \left[(r - r_2)\, a_n(r)\right]' \Big|_{r=r_2}$$
$$= \left[(r + \tfrac{1}{2})\, a_n(r)\right]' \Big|_{r=-\frac{1}{2}}$$

$$c_2 = \left[(r + \tfrac{1}{2})\, a_2(r)\right]' \Big|_{r=-\frac{1}{2}}$$
$$= \left((r + \tfrac{1}{2})\left(\frac{-a_0}{(r + \frac{3}{2})(r + \frac{5}{2})}\right)\right)' \Bigg|_{r=-\frac{1}{2}}$$
$$= \left(\frac{-r - \frac{1}{2}}{r^2 + 4r + 15/4}\right)' \Bigg|_{r=-\frac{1}{2}}$$
$$= \frac{-r^2 - 4r - 15/4 - (-r - \frac{1}{2})(2r + 4)}{(r^2 + 4r + 15/4)^2} \Bigg|_{r=-\frac{1}{2}}$$

$$= \frac{-(-\frac{1}{2})^2 - 4(-\frac{1}{2}) - \frac{15}{4}}{\left((-\frac{1}{2})^2 + 4(-\frac{1}{2}) + \frac{15}{4}\right)^2}$$

$$= \frac{-\frac{1}{4} + 2 - \frac{15}{4}}{(\frac{1}{4} - 2 + 15/4)^2}$$

$$= \frac{2-4}{(4-2)^2}$$

$$= \frac{-2}{4}$$

$$= \frac{-1}{2}$$

$$C_4 = \left[(r+\tfrac{1}{2})\, a_4(r)\right]' \Big|_{r=-\frac{1}{2}}$$

$$= \left((r+\tfrac{1}{2})'\, a_4(r) + (r+\tfrac{1}{2})\, a_4'(r)\right)\Big|_{r=-\frac{1}{2}}$$
$$= a_4(-1/2)$$
$$= \frac{-a_2(-1/2)}{(4-1)(4)}$$
$$= \frac{1}{(12)(2)}$$
$$= \frac{1}{24}$$

$$y_2 = x^{-1/2} - \frac{x^{3/2}}{2} + \frac{x^{5/2}}{24} - \ldots$$

$y_2$ is called the Bessel function of the second kind of order one-half. The standard notation is $Y_{\frac{1}{2}}(x)$.

E.g. Find 2 linearly independent solns of the Bessel eqn of order $\frac{11}{2}$.

Soln:

The Bessel eqn is $x^2y'' + xy' + (x^2 - \frac{121}{4})y = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $x^2y'' + xy' + x^2y - \frac{121}{4}y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

$$- \sum_{n=0}^{\infty} \frac{121}{4} a_n x^{n+r} = 0$$

Take $n=0$:

$a_0 (r)(r-1) + a_0 (r) - \frac{121}{4} a_0 = 0$

$a_0 (r^2 - r + r - \frac{121}{4}) = 0$

$r^2 - \frac{121}{4} = 0$

$r_1 = \frac{11}{2}, \quad r_2 = -\frac{11}{2}$

For $r = \frac{11}{2}$, take $n=1$:

$a_1 (1 + \frac{11}{2})(\frac{11}{2}) + a_1 (1 + \frac{11}{2}) - a_1 (\frac{121}{4}) = 0$

$a_1 ((1 + \frac{11}{2})(\frac{11}{2}) + (1 + \frac{11}{2}) - \frac{121}{4}) = 0$

$a_1 = 0$

Take $n \geq 2$:

$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - a_n\left(\frac{121}{4}\right) = 0$

$a_n\left((n+r)(n+r-1) + (n+r) - \frac{121}{4}\right) = -a_{n-2}$

Let $a = n+r$

$a(a-1) + a - \frac{121}{4}$

$a^2 - \frac{121}{4}$

$(a - \frac{11}{2})(a + \frac{11}{2})$

$(n+r - \frac{11}{2})(n+r + \frac{11}{2})$

$$a_n = \frac{-a_{n-2}}{(n+r-\frac{11}{2})(n+r+\frac{11}{2})}$$

For $r = \frac{11}{2}$, $a_n = \dfrac{-a_{n-2}}{(n)(n+11)}$

Since $a_1 = 0$, $a_{2k+1} = 0$.

| $n=2$: | $n=4$: | $n=6$: |
| --- | --- | --- |
| $a_2 = \dfrac{-a_0}{(2)(13)} = \dfrac{-1}{26}$ | $a_4 = \dfrac{-a_2}{(4)(15)} = \dfrac{1}{26 \cdot 50}$ | $a_6 = \dfrac{-a_4}{(6)(17)} = \dfrac{-1}{6 \cdot 17 \cdot 26 \cdot 50}$ |

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^{\frac{11}{2}} + a_2 x^{\frac{15}{2}} + a_4 x^{\frac{19}{2}} + \dots$$
$$= x^{11/2} - \frac{x^{15/2}}{26} + \frac{x^{19/2}}{26 \cdot 50} - \dots$$

$$y_2 = a\log(x)\,y_1 + x^{r_2}\left(1+\sum_{n=1}^{\infty} c_n(r)\,x^n\right)\Big|_{r=r_2}$$

$$a = \lim_{r\to r_2}(r-r_2)\,a_N(r), \quad N = r_1 - r_2 = \tfrac{11}{2} - \left(-\tfrac{11}{2}\right) = 11$$

$$= \lim_{r\to -\frac{11}{2}}\left(r+\tfrac{11}{2}\right)a_{11}(r)$$

$$= 0$$

$$c_n = \left[(r-r_2)\,a_n(r)\right]'\Big|_{r=r_2}$$

$$c_2 = \left[\left(r+\tfrac{11}{2}\right)a_2(r)\right]'\Big|_{r=-\frac{11}{2}}$$

$$= \left(\left(r+\tfrac{11}{2}\right)\left(\frac{-a_0}{\left(r-\frac{7}{2}\right)\left(r+\frac{15}{2}\right)}\right)\right)'\Bigg|_{r=-\frac{11}{2}}$$

$$= \left(\frac{-r-\frac{11}{2}}{r^2+4r-\frac{105}{4}}\right)'\Bigg|_{r=-\frac{11}{2}}$$

$$= \frac{(-1)\left(r^2+4r-\frac{105}{4}\right) - \left(-r-\frac{11}{2}\right)(2r+4)}{\left(r^2+4r-\frac{105}{4}\right)^2}\Bigg|_{r=-\frac{11}{2}}$$

$$= \frac{(-1)\left(\frac{121}{4}+4\left(-\frac{11}{2}\right)-\frac{105}{4}\right) - \left(+\frac{11}{2}-\frac{11}{2}\right)\left(2\left(-\frac{11}{2}\right)+4\right)}{\left(\left(-\frac{11}{2}\right)^2+4\left(-\frac{11}{2}\right)-\frac{105}{4}\right)^2}$$

$$= \frac{(-1)(4-22)}{(4-22)^2}$$

$$= \frac{18}{18^2}$$

$$= \frac{1}{18}$$

$$c_4 = [(r-r_2)a_4(r)]'\big|_{r=r_2}$$
$$= a_4(-\tfrac{11}{2})$$
$$= \frac{-a_2(-\frac{11}{2})}{(4-11)(4)}$$
$$= \frac{1}{(-7)(4)(-9)(2)}$$
$$= \frac{1}{504}$$

$$y_2 = x^{-11/2} + \frac{x^{-7/2}}{18} + \frac{x^{-3/2}}{504} + \dots$$

E.g. Find 2 linearly independent solns of the Bessel eqn of order 1.

Soln:

The Bessel eqn is $x^2y'' + xy' + (x^2-1)y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $x^2y'' + xy' + x^2y - y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

$$- \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Take $n=0$

$a_0(r)(r-1) + a_0(r) - a_0 = 0$

$a_0(r^2 - r + r - 1) = 0$

$r^2 = 1$

$r_1 = 1, \; r_2 = -1$

For $r=1$, take $n=1$

$a_1(2)(1) + a_1(2) - a_1 = 0$

$3a_1 = 0$

$a_1 = 0$

Take $n \geq 2$

$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - a_n = 0$

$a_n((n+r)(n+r-1) + (n+r) - 1) = -a_{n-2}$

Let $a = n+r$

$a(a-1) + a - 1$

$a^2 - 1$

$(a-1)(a+1)$

$(n+r-1)(n+r+1)$

$$a_n = \frac{-a_{n-2}}{(n+r-1)(n+r+1)}$$

For $r=1$, $a_n = \dfrac{-a_{n-2}}{n(n+2)}$

Since $a_1 = 0$, $a_{2k+1} = 0$.

| $n=2$: | $n=4$: | $n=6$: |
|---|---|---|
| $a_2 = \dfrac{-a_0}{2(4)}$ | $a_4 = \dfrac{-a_2}{4 \cdot 6}$ | $a_6 = \dfrac{-a_4}{6 \cdot 8}$ |
| $= \dfrac{-1}{8}$ | $= \dfrac{1}{192}$ | $= \dfrac{-1}{6 \cdot 8 \cdot 192}$ |

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x + a_2 x^3 + a_4 x^5 + a_6 x^7 + \ldots$$
$$= x - \frac{x^3}{8} + \frac{x^5}{192} - \frac{x^7}{6 \cdot 8 \cdot 192} + \ldots$$

$$y_2 = a \log(x) y_1 + x^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r) x^n\right)\Big|_{r=r_2}$$

$$a = \lim_{r \to r_2} (r - r_2) a_N(r), \quad N = r_1 - r_2$$
$$= 1-(-1)$$
$$= 2$$

$$= \lim_{r \to -1} (r+1) a_2(r)$$

$$= \lim_{r \to -1} (r+1)\left(\frac{-a_0}{(r+1)(r+3)}\right)$$

$$= \lim_{r \to -1} \frac{-1}{r+3}$$

$$= \frac{-1}{2}$$

$$c_n = \left[(r - r_2) a_n(r)\right]' \Big|_{r=r_2}$$

$$c_2 = \left[(r+1) a_2(r)\right]' \Big|_{r=-1}$$
$$= \left((r+1)\left(\frac{-a_0}{(r+1)(r+3)}\right)\right)' \Big|_{r=-1}$$
$$= \left(\frac{-1}{r+3}\right)' \Big|_{r=-1}$$
$$= \frac{1}{(r+3)^2} \Big|_{r=-1}$$
$$= \frac{1}{4}$$

$$C_4 = [(r+1)a_4(r)]' \Big|_{r=r_2}$$

$$= \left((r+1)\left(\frac{-a_2}{(4-2)4}\right)\right)' \Big|_{r=-1}$$

$$= \left((r+1)\left(\frac{1}{8(r+1)(r+3)}\right)\right)' \Big|_{r=-1}$$

$$= \left(\frac{1}{8r+24}\right)' \Big|_{r=-1}$$

$$= \frac{-8}{(8r+24)^2} \Big|_{r=-1}$$

$$= \frac{-8}{16^2}$$

$$= \frac{-1}{32}$$

$$y_2 = -\tfrac{1}{2}\log(x)y_1 + x^{-1} + \frac{x}{4} - \frac{x^3}{32} + \dots$$

E.g. Find 2 linearly independent solns of the Bessel eqn of order $\frac{3}{2}$.

Soln:

Bessel eqn: $x^2y'' + xy' + (x^2 - \frac{9}{4})y = 0$

We can rewrite it as

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r}$$

$$+ \sum_{n=0}^{\infty} -\tfrac{9}{4}a_n x^{n+r} = 0$$

Take $n=0$:

$a_0(r)(r-1) + a_0(r) - a_0(\frac{9}{4}) = 0$

$a_0(r^2 - r + r - \frac{9}{4}) = 0$

$r^2 = \frac{9}{4}$

$r_1 = \frac{3}{2}, \quad r_2 = -\frac{3}{2}$

For $r = \frac{3}{2}$, take $n=1$

$a_1(\frac{5}{2})(\frac{3}{2}) + a_1(\frac{3}{2}) - a_1(\frac{9}{4}) = 0$

$a_1(\frac{15}{4} + \frac{3}{2} - \frac{9}{4}) = 0$

$a_1 = 0$

Take $n \geq 2$:

$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - \frac{9}{4}a_n = 0$

$a_n((n+r)(n+r-1) + (n+r) - \frac{9}{4}) = -a_{n-2}$

Let $a = n+r$

$a(a-1) + a - \frac{9}{4}$

$a^2 - \frac{9}{4}$

$(a - \frac{3}{2})(a + \frac{3}{2})$

$(n+r-\frac{3}{2})(n+r+\frac{3}{2})$

$$a_n = \frac{-a_{n-2}}{(n+r-\frac{3}{2})(n+r+\frac{3}{2})}$$

For $r = \frac{3}{2}$, $a_n = \dfrac{-a_{n-2}}{(n)(n+3)}$

Since $a_1 = 0$, $a_{2k+1} = 0$

| $n=2$: | $n=4$: | $n=6$: |
|---|---|---|
| $a_2 = \frac{-a_0}{(2)(5)}$ | $a_4 = \frac{-a_2}{(4)(7)}$ | $a_6 = \frac{-a_4}{(6)(9)}$ |
| $= \frac{-1}{10}$ | $= \frac{1}{280}$ | $= \frac{-1}{54 \cdot 280}$ |

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^{3/2} + a_2 x^{7/2} + a_4 x^{11/2} + \ldots$$
$$= x^{3/2} - \frac{x^{7/2}}{10} + \frac{x^{11/2}}{280} - \ldots$$

$$r_1 - r_2 = \tfrac{3}{2} - \left(-\tfrac{3}{2}\right)$$
$$= 3$$

$N = 3$

$$y_2 = a \log(x) y_1 + x^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r) x^n\right) \Big|_{r=r_2}$$

$$a = \lim_{r \to r_2} (r - r_2) a_N(r)$$
$$= \lim_{r \to -\frac{3}{2}} \left(r + \tfrac{3}{2}\right) a_3(r)$$
$$= 0$$

$$c_n = \left[(r - r_2) a_n(r)\right]' \Big|_{r=r_2}$$

$$c_2 = \left[\left(r + \tfrac{3}{2}\right) a_2(r)\right]' \Big|_{r=-3/2}$$
$$= \left(\left(r + \tfrac{3}{2}\right)\left(\frac{-a_0(r)}{(n+r-3/2)(n+r+\frac{3}{2})}\right)\right)' \Bigg|_{r=-\frac{3}{2}}$$
$$= \frac{-1}{(2-3)(2)}$$
$$= \frac{1}{2}$$

$$y_2 = x^{-3/2} + \frac{x^{1/2}}{2} + \ldots$$